Chapter One:
Propositions, Possible Worlds, and Sets

1 What to Expect from Formal Logic

You already engage in “critical thinking.” The point of studying formal logic is to refine the way in which you do it.

In order to improve how you engage in critical thinking, you need to alter how you do it. So, this book introduces you to new terminology. It forces you to pay attention to distinctions that you most likely did not pay much attention to before. It makes use of ideas that will be quite foreign to you when they first come up. A lot of students recoil a bit at the whole enterprise. They feel that there is too much detail; that it’s too technical; that it’s too abstract; that it’s ultimately useless or irrelevant.

That reaction is completely understandable. But here’s the thing: the subtle distinctions and the new terminology and the attention to detail is kind of the whole point. Let me give you an analogy. A lot of you probably cook for yourselves or others at least occasionally. You probably know how to chop stuff with a knife. Now imagine that you went to a cooking class and they just told you: hold the knife in your dominant hand, move it up and down, and make sure to keep your other hand out of the way. Obviously, that wouldn’t be very useful. But now imagine that you went to the cooking class and they told you: exactly where to place your fingers on the knife when you are holding it, and exactly where to position your other hand as you chop; how to stand so that you do not get tired while you are chopping; what the difference is between cubing, dicing, and juliennning something, and how to do each of those things; what knife is appropriate for which tasks; different techniques for sharpening knives and how often to sharpen them; and so on. The second class would be a lot more useful precisely because of all the new details that you would be introduced to and all the intricate skills that you would be forced to practice. Anyone can chop up some vegetables, but not anyone can perfectly dice an onion or julienne a bell pepper. It’s only when you start pushing the boundaries of what everyone already knows that you start learning something valuable.

So, while you are working your way through this book, I encourage you to see yourself as pushing that boundary. There are a lot of details to master. There are a lot of new ideas to get used to. There is something foreign and unusual about what you are being exposed to. You are going to have to struggle through
some unfamiliar terrain. But that is precisely what makes it worthwhile. You are advancing beyond how people normally engage in critical thinking. You are learning to do it in a much more systematic and sophisticated way. If it weren’t new and different and challenging, then you wouldn’t be learning anything.

2 Sentences Vs. Propositions

I want to start by introducing a distinction between sentences, on the one hand, and propositions, on the other. Throughout this book, we are going to build up a kind of overall conceptual framework for improving our critical thinking. Starting to get clear on the distinction between sentences and propositions is the first step in building up that framework.

So what is the difference between a sentence and a proposition? Sentences are bits of language. They are made up of words. We utter them when we have conversations with each other. It may seem strange to think of it in this way, but when you talk on the phone with someone, you utter certain sentences, and then the other person utters more sentences in response. If you were to write a letter to someone, then you would write various sentences down on a piece of paper. You are looking at various sentences right now as you are reading this paragraph.

What is a proposition? Suppose that you say to your friend, “The bus arrives at noon.” Distinguish between two things that happen. One is that you utter a certain sentence—a certain string of words. Namely, you utter the word ‘The’, followed by the word ‘bus’, followed by the word ‘arrives’, and so on. But, second, you also convey a certain piece of information. You tell your friend when the bus arrives.

Note that you could have expressed or communicated the very same piece of information in a different way. For instance, you could have said, “The bus arrives at twelve pm.” In that case, you would have uttered a slightly different string of words. For you would have uttered the word ‘twelve’, which is not a word that we imagined you saying originally. But, although you would have uttered a slightly different sentence—a slightly different string of words—had you said, “The bus arrives at twelve pm,” you still would have expressed exactly the same information.

So we can distinguish between the piece of information that you communicated to your friend—namely, that the bus arrives at noon—and the sentence that you chose to use in order to convey that piece of information. There are a number of different sentences that you could have used in order to express that one piece of information. The piece of information in question, which you communicated to your friend by saying, “The bus arrives at noon,” but which you could have communicated in any number of different ways, is what we call a proposition. So propositions, as we are going to use that term, are not sentences, but are, rather,
the individual pieces of information that we communicate to each other by uttering sentences.

Note that we may also convey information to each other without uttering any sentences at all. For example, suppose that you ask me whether I want to go see *Jurassic Park*, and I give you a thumbs up. In that case, I did not say any words. I simply made a gesture. But I did manage to communicate to you a certain piece of information, namely, that I want to see the movie. The piece of information that I communicated to you—namely, that I want to see the movie—is what we call a proposition.

A good way to think about the distinction between sentences and propositions is that propositions are what we express to each other when we communicate. They are the little bits of information—the individual messages—that we pass back and forth. Sentences, on the other hand, are the means by which we convey those messages or pieces of information to each other, at least in most cases. Sometimes, as in the example I just gave, we convey information in some other way, by, say, nodding our head or giving a thumbs up.

Think about people who speak different languages from ours. The words and sentences they utter are different. But that just means that they are employing a different means of communication. They can still convey exactly the same information. They can still pass on the same messages. So again we can make out a distinction between sentences, on the one hand, which differ from language to language, and the pieces of information that we use sentences to communicate to each other. People speaking different languages, and thus employing different sentences, can still pass on the same information. Propositions are the messages, or pieces of information, that get passed on.

So suppose that I say, “John is in the library,” for example. We want to distinguish between two things. First, there is the sentence that I utter, which is ‘John is in the library’. Were I to have instead said, “John está en la biblioteca,” then I would have uttered a completely different sentence. Or suppose that instead of saying, “John is in the library,” I had instead said, “John is currently in the library,” or, “John’s present location is the library.” Again, in either case, I would have uttered a slightly different sentence.

Second, there is the piece of information that I communicate or pass on when I say, “John is in the library.” The information that I pass on is that John is in the library. Had I said, “John está en la biblioteca,” or, “John is currently in the library,” or, “John’s present location is the library,” I still would have communicated that very same piece of information, although I would have been using a different sentence in order to do so.

Another way to think about the distinction between sentences and propositions is that sentences are devices that we use in order to make claims about the world.
You utter a certain string of words, such as ‘The Red Sox will win exactly one World Series in the rest of this decade’, and, as a result, you make a certain claim about the world. You thereby take a stand on what is going to happen. The specific claim about the world that you made—namely, that the Red Sox will win exactly one World Series over the rest of this decade—is a proposition. Note that you could have made the same claim about the world by uttering a different sentence, such as the slightly different ‘The Red Sox will win one, and only one, World Series over the rest of this decade’. So a sentence is just a device that we use to make claims about the world. Propositions are the claims that we make about the world, which can be expressed or put forward in a number of different ways.

Do Practice Questions A
(all practice questions located at the end of the chapters)

3 The Things That are True/False

There are two truth-values: true and false. Sometimes people act as if sentences have truth-values. So, for instance, someone might say that ‘The Grand Canyon is majestic’ is a true sentence. But it is problematic to think of sentences as having truth-values, that is, of being either true or false. It is much better to see propositions—that is, the claims about the world that we put forward or express by uttering sentences—as the things which are true or false.

Let me give an example to indicate why. Take the sentence ‘Obama is the President’. Now suppose we ask: is that sentence true or false? Well, you might say, obviously Obama is the President. We all know that. So, you might say, the sentence in question is true. But now suppose someone says: what about all the other people in the world named ‘Obama’? First of all, there is Michelle Obama, and then there are Sasha and Malia Obama—Barack Obama’s daughters—and presumably a whole bunch of other people too, none of whom are the President. So, it seems like we can’t say that the sentence ‘Obama is the President’ is true after all, because of all those people. But it doesn’t seem right to say that it is false either, because Barack Obama is indeed the President. So now what do we say? Do we say that the sentence ‘Obama is the President’ is neither true nor false?

A better option is to forego talk of whether sentences are true or false altogether. Sentences are just not the right sort of thing to have truth-values. So,
rather than trying to decide whether the sentence ‘Obama is the President’ is true or false, here is what we say.

There is this one sentence ‘Obama is the President’. Depending on the context, however, that sentence might be used to express a variety of different claims about the world. That is, it may be used to express a number of different propositions. It is those propositions that we assess for truth and falsity. Not the sentence itself.

So, for example, by far the most salient person named ‘Obama’, at the current moment, is Barack Obama. Thus, when, in the course of a normal conversation, someone simply says, “Obama is the President,” we assume that she is talking about Barack Obama. We assume that she is saying that he is the President. In other words, the proposition or piece of information that we take her to be conveying is that Barack Obama is the President. That proposition—that Barack Obama is the President—is true.

But we can also imagine a different context where someone else with the name ‘Obama’, such as Sasha Obama, is being discussed. If the speaker makes clear that she means to be referring to Sasha Obama when she uses the name ‘Obama’, and she then says, “Obama is the President,” then she would thereby make the claim, not that Barack Obama is the President, but that Sasha Obama is the President. That latter proposition—that Sasha Obama is the President—is false.

So we have one sentence—namely, ‘Obama is the President’—and a variety of different claims about the world, or propositions, as we call them, which that sentence can be used to express. For example, that Barack Obama is the President, that Sasha Obama is the President, that Malia Obama is the President, and so on. Instead of talking about whether the sentence ‘Obama is the President’ is true or false, we talk about whether the various propositions that it can be used to express—that is, the various claims about the world that it can be used to make—are true or false.

So it is propositions, and not sentences, which are best thought of as being the bearers of truth and falsity. Sentences are just a means of communication. We use them to pass on messages and information. We use them to make claims about the world. It is the things that get passed on—that is, the messages or pieces of information that get communicated, the claims that get made—which are either true or false.

4 Possible Worlds

In some cases, it is clear that two sentences express the very same proposition. For instance, consider these two sentences: first, ‘Neither Frank nor Maria was killed by a zombie’; second ‘Frank was not killed by a zombie and Maria was not killed by a zombie’. The proposition—the claim about the world—that the first sentence
expresses is identical to the proposition that the second sentence expresses. They clearly express or put forward the exact same piece of information.

In other cases, it is clear that two sentences do not express the same proposition. For instance, the sentence ‘Colin plays basketball’ obviously does not express the very same proposition as the sentence ‘Colin coaches basketball’. The two sentences clearly make different claims about Colin.

What about cases where we are not sure whether two sentences, or two utterances, express the very same proposition? So, for example, consider two brothers, Click and Clack. Suppose that Click says, “My brother is a bad driver,” and then Clack says, “My brother is a bad driver.” In that case, both Click and Clack utter the very same sentence. But do they express the same proposition? Do they each put forward the same claim about the world?

I eventually want to introduce a strategy that we can use to help determine whether two propositions—such as the proposition expressed by Click and the proposition expressed by Clack—are the same or different. Explaining this strategy, however, requires first introducing the notion of a possible world.

First observe, and I take this to be completely uncontroversial, that there is a way that the world—by which I mean the entire universe, not just our planet—really is. So, for example, Barack Obama is the President of the US; the city of Cambridge, MA has a mayor; in the 1860s, there was a civil war in the United States; there are a number of mountains, including Mt. Everest, which are over 20,000 feet tall; a number of different languages are spoken in Europe; there are a huge number of different stars and planets; and so on.

Now, it would be impossible to describe in complete detail exactly how the world really is—all I did above was mention a handful of facts about what the world is like. And there is significant disagreement about exactly what the world is like. Much of that disagreement may persist forever. But surely there is a way that the world really is, whether we can come to agreement about all aspects of it or not.

There are also ways that the world might conceivably have turned out to be, even though it did not in fact turn out that way. For example, Hillary Clinton did not in fact win the Democratic primary in 2008. But she could have: the world could have turned out much like it actually did, except with Hillary Clinton winning the Democratic primary in 2008, rather than Barack Obama. The San Francisco 49ers, unfortunately, did not with the 2013 Super Bowl. But they could have: the world could have turned out much like it actually did, except with the 49ers scoring a touchdown on their final drive, and then holding on to win the Super Bowl, instead of failing to score a touchdown and then losing, which is what actually happened.

Now, neither of the two alternative possibilities that I have just mentioned
are very “far out” possibilities. The world did not in fact turn out in either of the ways that I described, but it easily could have. We can also consider more radical possibilities. For example, there is a sense in which even I could have won the Democratic primary in 2008. In order for that to have happened, the world would have needed to be a lot different in all sorts of ways, but it is still at least conceivable or coherently imaginable for the world to have turned out that way, rather than how it actually turned out. We can at least make sense of an alternative, fantastical scenario where I win the 2008 Democratic primary, rather than Barack Obama, even though none of us takes that possibility seriously as something that might really have happened. Similarly, the world could have contained only two planets, rather than the massive amounts of planets that it actually contains. In fact, it seems to me, the world could have contained only a single atom, floating around in the void. Those are both very “far out” possibilities. But the world could have conceivably turned out in either of those ways: the world’s having turned out in either of those ways, rather than in the way it actually did, is at least coherently imaginable. By contrast, we cannot even coherently imagine the world’s turning out in such a way that I am both exactly 5’10” tall and exactly 5’11” tall.

So there is a way that the world really is, and there are a whole bunch of different ways that the world could have conceivably turned out to be, even though it did not in fact turn out in any of those ways. I take this much to be obvious.

Now here is where possible worlds come in. For every single way that the world could conceivably have turned out to be, we postulate a possible world that is that way. For example, as I noted before, the world could have been very much like it actually is, except with Hillary Clinton winning the Democratic primary in 2008. So we postulate a possible world that is that way. That is, we postulate a possible world that is very much like how the world actually is, except with Hillary Clinton winning the Democratic primary in 2008, rather than Barack Obama. We said before that the world could, conceivably, have contained just two planets. So we postulate a possible world that is that way.

I encourage you not to think about these possible worlds as alternate universes, or distant places that we might travel to in a space ship. If you think of possible worlds in that way, then postulating or supposing that there are all these various different possible worlds is going to seem crazy. Instead, think about possible worlds more abstractly.

Let me give an analogy. Think of a number line. A number line consists of a whole bunch of points lined up in a series. Each point stands for or represents a number. The point in the middle stands for zero, while the points to the left of that one stand for numbers less than zero, and the points to the right of it stand for numbers greater than zero. There are an infinite number of points on
the number line, each one corresponding to a different number.

Now suppose that someone asks you: Where is this number line? Where are all the various points? What are they made out of? Can I see them or touch them? If not, why do you think that they exist? I take it that this would be a bizarre series of questions. A number line is just a tool that we use in order to help us think about numbers. It’s existence does not have to be justified or defended. It’s not even clear what it would mean to defend the existence of the number line.

Think of possible worlds as akin to the points on a number line. There are a vast quantity of them. In particular, for every way that the world could conceivably have turned out to be, there is, we suppose, a possible world that is that way. So there is one possible world which is the way that the world really is. At that world, Barack Obama is the President of the US; the city of Cambridge, MA has a mayor; in the 1860s, there was a civil war in the United States; there are a number of mountains, including Mt. Everest, which are over 20,000 feet tall; a number of different languages are spoken in Europe; there are a huge number of different stars and planets; and so on. We call that one possible world—the one which matches up exactly with how the world actually is—the actual world. All the other possible worlds differ in some way from that one.

But these possible worlds are not alternative universes. They are just tools that we use in order to help us think about different possibilities. Asking why we should think that there are all these different possible worlds is like asking why we should think that the various points on the number line really exist. The different points just stand for different numbers. There is no question of whether the points “really exist” or not. Similarly, the different possible worlds just stand for or represent different ways that the world might have turned out. There is no question about whether they “really exist” or not. We all agree that the world could have conceivably turned out differently in any number of ways, both large and small. Postulating the existence of possible worlds, to stand for those different possibilities, just helps us think more clearly and precisely about all the different ways that the world might have turned out.

So from now on we will unapologetically suppose that, for every single way that the world could conceivably have turned out to be, including the way that it actually did turn out to be, there is a possible world that is that way. We will further suppose that there are no other possible worlds. This means that, while there are a vast quantity of different possible worlds, there is also some limit on what possible worlds there are. For example, there is no possible world where I am both exactly 5’10” tall and exactly 5’11” tall. For, as I mentioned before, the world could not conceivably have turned out that way. We only allow possible worlds which stand for or represent genuine possibilities.
5 Truth-Value at a World

We have now introduced perhaps the two most important notions that we will rely on throughout the rest of this book: the notion of a proposition and the notion of a possible world. We are going to spend some more time trying to make sure that we understand these two notions as well as we can. To go back to an analogy that I made earlier: we are still in the sharpening-our-knives portion of the process. We’ve got to spend some time perfecting our basic techniques, as it were, before we can move on to more ambitious projects.

So consider some proposition, or claim about the world, for example: that Barack Obama won the Democratic primary in 2008. That proposition—that Barack Obama won the Democratic primary in 2008—is true or false at any given possible world, depending on what that world is like.

First consider the actual world. At the actual world, Barack Obama did indeed win the Democratic primary in 2008. So the proposition or claim that Barack Obama won the Democratic primary in 2008 is true at the actual world. Let’s let:

- @ = the actual world
- p = that Barack Obama won the Democratic primary in 2008

Then we may represent the fact that it is true at the actual world that Barack Obama won the Democratic primary in 2008 with this little table:

<table>
<thead>
<tr>
<th></th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>@</td>
<td>T</td>
</tr>
</tbody>
</table>

Now consider some possible world where Hillary Clinton wins the Democratic primary in 2008, rather than Barack Obama. Note that there are a whole bunch of different possible worlds at which Hillary Clinton wins the 2008 Democratic primary rather than Obama: there is a possible world at which Hillary Clinton wins that race, and then goes on to win the presidential election; there is a possible world at which she wins the 2008 Democratic primary, but then goes on to lose the presidential election to John McCain; there is a possible world at which she wins the 2008 Democratic primary, but then loses the presidential election to Mitt Romney; there is a possible world at which she wins the 2008 primary, but then gets assassinated before the presidential election; there is a possible world at which she wins the Democratic primary in 2008, but then World War III starts and no presidential election is even held; and so on and so forth. Each of those is a different, coherently imaginable way for the world to have turned out.

The proposition that Barack Obama won the Democratic primary in 2008 is false at any world where Hillary Clinton wins that race rather than him. So let’s let:
• $w_1 =$ some possible world where Hillary Clinton wins the Democratic primary in 2008, rather than Barack Obama

Then we may represent the fact that it is false at world $w_1$ that Barack Obama won the Democratic primary in 2008 by adding on to our table as follows:

\[
\begin{array}{cc}
\emptyset & p \\
@ & T \\
w_1 & F \\
\end{array}
\]

We may keep considering different possible worlds. The proposition that Barack Obama won the Democratic primary in 2008 is either going to be true or false at each one, depending on whether Obama won the 2008 Democratic primary at that world or not. So if we went through all the various possible worlds, we would eventually end up with a table looking something like this:

\[
\begin{array}{cc}
\emptyset & p \\
@ & T \\
w_1 & F \\
w_2 & F \\
w_3 & T \\
w_4 & T \\
w_5 & F \\
\vdots & \vdots \\
\end{array}
\]

Note that we could do the same thing for any proposition whatsoever. So, for instance, consider the proposition that Boston, MA is the capital of the United States. That proposition is true or false at any given possible world depending on whether, at that world, Boston is the capital of the US or not. So, at the actual world, it is false that Boston is the US capital. But we can at least coherently imagine the world having turned out in such a way that Boston was the capital. For instance, we could imagine history proceeding in much the way that it actually did, with the exception that Washington D.C. gets destroyed during the course of the Civil War, and then a decision is made to move the US capital to Boston. At a possible world like that, it is true that Boston is the capital of the US.

Do Practice Questions B
6 The Distinctness Principle

Now that we have introduced the idea of a possible world, let’s go back to a question that I posed earlier. In some cases, I said, it is clear that two sentences express the very same proposition—that is, make the very same claim about the world. In other cases, it is clear that two sentences do not express the same proposition. But what about cases where we are not sure?

The example I gave was this: consider two brothers, Click and Clack. Suppose that Click says, “My brother is a bad driver,” and then Clack says, “My brother is a bad driver.” In that case, both Click and Clack utter the very same sentence. But do they express the same proposition? Do they each put forward the same claim about the world?

Here is a strategy that we can use to figure out the answer. We know that when Click says, “My brother is a bad driver,” there is some claim that he manages to make about the world. In other words, there is some proposition that he expresses. Let’s let:

• \( p \) = the proposition or claim about the world that Click expresses when he says, “My brother is a bad driver.”

Similarly, we know that when Clack says, “My brother is a bad driver,” there is some proposition that he expresses, or some claim that he makes. Let’s let:

• \( q \) = the proposition or claim about the world that Clack expresses when he says, “My brother is a bad driver.”

Our situation is that we are not entirely sure whether \( p \) and \( q \) are the same. So we want to know more about each of those propositions. We want to better identify each one, so that we can tell whether they are the same or different.

One way to get a clearer picture of what propositions we are dealing with is by considering how each proposition fits with different possible scenarios, or different possible worlds. So, for example, consider a possible world where Click has a spotless driving record and is an excellent overall driver, while Clack has incurred numerous tickets and fines and has been in a number of accidents that were all his fault. Let’s let:

• \( w \) = a possible world where Click has a spotless driving record and is an excellent overall driver, while Clack has incurred numerous tickets and fines and has been in a number of accidents that were all his fault.

Now we may ask, of each of our two propositions \( p \) and \( q \), whether they are true or false at \( w \).
Let’s first consider \( p \). This is the proposition or claim that Click puts forward when he says, “My brother is a bad driver.” Is that proposition true or false at \( w \)? Since Clack is obviously a terrible driver at world \( w \), it is clear that \( p \)—the proposition that Click puts forward when he says, “My brother is a bad driver”—is true at world \( w \). At other worlds, such as worlds where Clack is a marvelous driver who has never even been close to being an accident, \( p \) is false. But it is obviously true at \( w \).

Now let’s consider \( q \). This is the proposition or claim that Clack expresses when he says, “My brother is a bad driver.” Is that proposition true or false at \( w \)? Since Click is an excellent driver at that world, it is clear that \( q \)—the proposition that Clack puts forward when he says, “My brother is a bad driver”—is false at world \( w \). Again, at other worlds, such as worlds where Click has hit multiple parked cars, and run over a couple of pedestrians, and consistently runs red lights, and so on, \( q \) is true. But \( q \) is clearly false at world \( w \).

So we have:

\[
\begin{array}{c|c|c}
\text{world } w & p & q \\
\hline
w & T & F \\
\end{array}
\]

Since \( p \) and \( q \) have different truth-values at world \( w \), they must be different propositions. Obviously, if \( p \) and \( q \) were identical to each other—if they were just one and the same proposition, or claim about the world—then they would have the same truth-value at \( w \). Indeed, they would then have the same truth-value at every possible world, since they would just be one and the same proposition. They cannot have different truth-values at some possible world and yet still be identical to each other. Let’s call this The Distinctness Principle:

**The Distinctness Principle:** If there is some possible world at which \( p \) and \( q \) differ in truth-value, then \( p \neq q \).

So, although Click and Clack both utter the same sentence, namely, ‘My brother is a bad driver’, each expresses a different proposition, or makes a different claim about the world. Intuitively, this makes sense. After all, when Click says, “My brother is a bad driver,” he is making a claim about Clack, whereas when Clack says, “My brother is a bad driver,” he is making a claim about Click. So, when you think about it, it seems obviously right that the two brothers put forward different propositions, or different claims about the world. Looking at the truth-value of their respective claims across various different possible worlds just helps to bring out even more clearly that the two brothers each expressed a different proposition.

**Do Practice Questions C**

12
The Proposition Expressed Varies with the Context

Some of the examples that we have just been looking at bring out a point that is worth mentioning: what proposition a sentence expresses—in other words, what claim it makes about the world—depends on the surrounding context. That is, it depends on things like who is speaking, who is being spoken to, where the utterance is taking place, when it is taking place, and so on.

So, for example, take the sentence ‘I’m hungry’. There is no such thing as the proposition or claim that this sentence puts forward. If Greg utters that sentence—if he is the speaker—then one proposition is expressed. On the other hand, if Isaac is the speaker—if he is the one who says, “I’m hungry”—then a different proposition is expressed.

Similarly, there is no such thing as the proposition that is expressed by the sentence ‘There is gold buried underneath this building’. If someone standing in Dugan Hall utters that sentence, then one proposition is put forward, whereas if someone standing in Coburn Hall utters that sentence, then an entirely different proposition is expressed. So what claim or proposition is expressed by the sentence ‘There is gold buried underneath this building’ depends on where the speaker is when she utters that sentence.

Or, to take one last example, consider the sentence ‘The Red Sox won the World Series last year’. If someone utters that sentence in 2005, she makes an entirely different claim about the world than someone who utters that sentence in 2010. So exactly what proposition is expressed by the sentence ‘The Red Sox won the World Series last year’ depends on when in time the person uttering the sentence is speaking.

So the overall lesson is that what proposition or claim a sentence expresses depends on the surrounding context. To put it a different way: you can take one sentence, and depending on what context is surrounding it, it may express different propositions, or put forward different claims about the world. We can represent that fact with the following diagram:
8 The Sameness Principle

Everybody would agree with the Distinctness Principle:

**The Distinctness Principle**: If there is some possible world at which $p$ and $q$ differ in truth-value, then $p \neq q$.

But what if $p$ and $q$ never differ in their truth-value? That is, what if their truth-value, at every single possible world, is always a match? Can we then conclude that $p = q$? There are some reasons for saying yes. That is, there are some reasons for also accepting the Sameness Principle:

**The Sameness Principle**: If there is no possible world at which $p$ and $q$ differ in truth-value, then $p = q$.

One way to motivate the Sameness Principle is by looking at examples. Consider the sentence ‘Neither Frank nor Maria was killed by a zombie’ and the sentence ‘Frank was not killed by a zombie and Maria was not killed by a zombie’. Each sentence puts forward a certain claim about the world. That is, each sentence expresses a proposition. Let:

- $p =$ the proposition or claim that is expressed by the sentence ‘Neither Frank nor Maria was killed by a zombie’
- $q =$ the proposition that is expressed by the sentence ‘Frank was not killed by a zombie and Maria was not killed by a zombie’

Note that, no matter what possible world we focus on, $p$ and $q$ are always going to have the same truth-value as each other. For instance, consider a world where Frank manages to escape the zombie invasion unharmed, but Maria is hunted down and killed by the zombies. At a world like that, $p$ is obviously false, and the same goes with $q$. Or consider a world where both Frank and Maria manage to escape from the zombie invaders. At a world like that, $p$ is clearly true and so is $q$.

Since $p$ and $q$ have the same truth-value at every possible world, the Sameness Principle implies that they are in fact one and the same proposition. So it implies that the two sentences—‘Neither Frank nor Maria was killed by a zombie’ and ‘Frank was not killed by a zombie and Maria was not killed by a zombie’—in fact express the same proposition, or make the very same claim about the world. On the face of it, that is exactly the right result. So the Sameness Principle generates the right result in this case. It seems to generate the right results in a number of other cases as well.

Now, in fact the Sameness Principle is rather controversial. In some cases, it seems to generate the wrong results. But, from now on, we are going to assume that it is true.
9 The Principle of Propositional Identity

Put together, The Distinctness Principle and The Sameness Principle give us this result:

The Principle of Propositional Identity: \( p = q \) if and only if there is no possible world at which \( p \) and \( q \) differ in their truth-value.

How do the The Distinctness Principle and The Sameness Principle combine to give us this result?

This is a good time to talk about the phrase ‘if and only if’. We will use this phrase quite often, so I want to make sure that we are all on the same page about what it means.

Consider this claim:

Hillary Clinton will win the 2016 presidential election if and only if she wins all 50 states.

This claim asserts two things. The first thing that it asserts:

Hillary Clinton will win the 2016 presidential election if she wins all 50 states.

In other words: as long as Hillary Clinton wins all 50 states, she will win the 2016 presidential election. This claim is obviously true. If she wins all 50 states, then not only will Clinton win the election, she will thereby win the biggest landslide victory in history.

The second thing that it asserts:

Hillary Clinton will win the 2016 presidential election only if she wins all 50 states.

In other words, her winning all 50 states is a requirement of her winning the 2016 election. Or, to put it yet another way: if she does not win all 50 states, then she will not win the election. This claim is obviously false. Clinton could easily win the election without winning all 50 states. For instance, if she won 49 states, or 48 states, or 47 states, or 46 states—those would all be other ways for her to win the election.

Let me give another example. Consider this claim:

A person is eligible to vote in a US presidential election if and only if she is 18 or over.
Again, this claim asserts two things. The first thing that it asserts is:

A person is eligible to vote in a US presidential election if she is 18 or over.

In other words, as long as someone is over 18 or over, she is eligible to vote in a US presidential election. This claim is false. Non-citizens are not allowed to vote. Convicted felons are not allowed to vote. Being a person 18 or over is not sufficient for being eligible to vote in a US presidential election.

The second thing that it asserts is:

A person is eligible to vote in a US presidential election only if she is 18 or over.

In other words, being 18 or over is a requirement—it is a necessary prerequisite—of being eligible to vote in a US presidential election. This claim is true.

So let us now return to:

The Principle of Propositional Identity: \( p = q \) if and only if there is no possible world at which \( p \) and \( q \) differ in their truth-value.

This claim asserts two things. The first thing that it asserts is:

\[ p = q \text{ if there is no possible world at which } p \text{ and } q \text{ differ in their truth-value.} \]

In other words, as long as there is no possible world at which \( p \) and \( q \) differ in their truth-value, \( p = q \). Note that this is exactly what The Sameness Principle says.

The second thing that The Principle of Propositional Identity asserts is:

\[ p = q \text{ only if there is no possible world at which } p \text{ and } q \text{ differ in their truth-value.} \]

In other words: for it to be true that \( p = q \), a necessary requirement is that there be no possible world at which \( p \) and \( q \) differ in truth-value. That is: if there is some possible world at which \( p \) and \( q \) differ in their truth-value, then \( p \neq q \). This is exactly what The Distinctness Principle says.

So The Principle of Propositional Identity does not say anything more than what the The Sameness Principle and The Distinctness Principle say, when combined. It just asserts both of those principles at once.

We will assume that The Principle of Propositional Identity is correct.

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Do Practice Questions D
10 Sets

One last notion that I want to introduce in this chapter is that of a set. Intuitively, a set is just a group of things. An example is: \{Tom Brady, Paul Pierce, David Ortiz\}. The things in a set are called its members. So the members of the set \{Tom Brady, Paul Pierce, David Ortiz\} are Tom Brady, Paul Pierce, and David Ortiz.

To say that \(x\) is a member of some set \(S\), we write: \(x \in S\). So, for instance, to say that Tom Brady is a member of the set \{Tom Brady, Paul Piece, David Ortiz\} we write: \(Tom Brady \in \{Tom Brady, Paul Piece, David Ortiz\}\).

To say that \(x\) is not a member of some set \(S\), we simply write: \(x \notin S\). So, for instance, to say that Wes Welker is not a member of the set \{Tom Brady, Paul Pierce, David Ortiz\}, we write: \(Wes Welker \notin \{Tom Brady, Paul Pierce, David Ortiz\}\).

A set can have any kind of thing as one of its members, including other sets. So the set \{Tom Brady, Paul Pierce, David Ortiz\} has as its members three athletes that are particularly well-loved by New Englanders. The set \{1,2,3,4\} has the numbers 1, 2, 3, and 4 as its members. But we may also consider sets which have propositions or possible worlds as their members. For instance, here is a set which has two propositions as its members: \{that Barack Obama is president, that Hillary Clinton is vice president\}. Here is a set which has the actual world as its one and only member: \{@\}. In this course, it is sets of propositions and sets of possible worlds that we will be most interested in.

10.1 Set Identity

A set is defined by its members. That is, sets \(S_1\) and \(S_2\) are identical if, and only if, they have exactly the same members.

A set \(S_1 = \) a set \(S_2\) if, and only if, \(S_1\) and \(S_2\) have exactly the same members.

So, for instance, the sets \{Tom Brady, Paul Pierce, David Ortiz\} and \{Paul Pierce, Tom Brady, David Ortiz\} are considered to be identical sets, since they have exactly the same members. Note that it does not matter in what order we list the members of a set. As long as exactly the same things are listed in two sets, the sets are identical.

10.2 Subsets

One set \(S_1\) is a subset of another set \(S_2\) just in case all of the members of the first set are also members of the second set. So, for instance, consider these two sets of
possible worlds:

\[ A = \{w_1, w_2, w_3, w_4\} \]
\[ B = \{w_1, w_2\} \]

Since all of the members of \( B \) are also members of \( A \), \( B \) counts as a subset of \( A \).

Here are some other subsets of \( A \):

\[ C = \{w_1\} \]
\[ D = \{w_1, w_2, w_3\} \]
\[ E = \{w_3, w_4\} \]
\[ F = \{w_2, w_3\} \]

Note that the following two sets are not subsets of \( A \):

\[ G = \{w_1, w_2, w_3, w_4, w_5\} \]
\[ H = \{w_1, w_6\} \]

There are some members of \( G \) that are not also members of \( A \). In particular, \( w_5 \in G \), but \( w_5 \notin A \). So \( G \) is not simply a subset of \( A \). The same goes with \( H \): some of its members are not members of \( A \). To be a subset of \( A \), all of its members would also have to be members of \( A \).

To say that \( S_1 \) is a subset of \( S_2 \) we write: \( S_1 \subseteq S_2 \).

\( S_1 \) is a subset of \( S_2 \) (in symbols: \( S_1 \subseteq S_2 \)) if and only if everything that is a member of \( S_1 \) is also a member of \( S_2 \).

Note that, by this definition, the set \( A \) is technically a “subset” of itself. For all of the members of \( A \), namely, \( w_1, w_2, w_3, \) and \( w_4 \), are, trivially, also members of \( A \). So, by our definition, \( A \subseteq A \). Indeed, by our definition, every set is a subset of itself. Sets like \( B, C, D, E, \) and \( F \) are known as proper subsets of \( A \), since they are subsets of \( A \) without being identical to \( A \) itself. In general, \( S_1 \) is a proper subset of \( S_2 \) just in case \( S_1 \) is a subset of \( S_2 \) without being identical to \( S_2 \) itself. To say that \( S_1 \) is a proper subset of \( S_2 \) we write: \( S_1 \subset S_2 \).
\( S_1 \) is a proper subset of \( S_2 \) (in symbols: \( S_1 \subset S_2 \)) if and only if \( S_1 \subseteq S_2 \), but \( S_1 \neq S_2 \).

10.3 The Null Set

There is, by convention, one set with no members. It is called the empty set or the null set. It is symbolized like this: \( \emptyset \).

\[ \emptyset = \text{the null set} = \text{the set that has no members} \]

10.4 Union

The union of a set \( S_1 \) with a set \( S_2 \) is itself a further set. It is the set that you get when you combine, or unite, all of the members \( S_1 \) with all of the members of \( S_2 \), and then stop. So, for instance, consider these two sets of possible worlds:

\[ M = \{ w_1, w_2, w_3 \} \]
\[ K = \{ w_3, w_4 \} \]

Then the union of \( M \) with \( K \) = the set that you get when you combine all of the members of the set \( M \) with all of the members of the set \( K \), and then stop = the set \( \{ w_1, w_2, w_3, w_4, w_5 \} \).

To refer to the union of a set \( S_1 \) with a set \( S_2 \) we write: \( (S_1 \cup S_2) \). We will sometimes leave off the parentheses for the sake of convenience. So, for instance, officially, to refer to the union of the set \( M \) with the set \( K \) we would write: \( (M \cup K) \). But often, as long as there is no chance of ambiguity or confusion, we will simply write: \( M \cup K \).

\((S_1 \cup S_2) = \text{the union of } S_1 \text{ with } S_2 = \text{the set that results when you combine, or unite, all of the members of } S_1 \text{ with all of the members of } S_2, \text{ and then stop. That is, by definition:} \]

- if \( x \in S_1 \), then \( x \in (S_1 \cup S_2) \)
- if \( x \in S_2 \), then \( x \in (S_1 \cup S_2) \)
- Otherwise, \( x \notin (S_1 \cup S_2) \)
So, to look at one more example, consider these two sets:

\[ \mathbb{B} = \{ w_1, w_2, w_3 \} \]

\[ \mathbb{C} = \{ w_3, w_4, w_5, w_6 \} \]

Now consider the union of those two sets: \( (\mathbb{B} \cup \mathbb{C}) \). \( w_1, w_2, \) and \( w_3 \) are all members of \( \mathbb{B} \). So, by definition, those three possible worlds are all members of \( (\mathbb{B} \cup \mathbb{C}) \) too. For, by definition, any member of \( \mathbb{B} \) also goes into the set \( (\mathbb{B} \cup \mathbb{C}) \). Thus, we know that \( (\mathbb{B} \cup \mathbb{C}) \) at least contains these three worlds:

\[ (\mathbb{B} \cup \mathbb{C}) = \{ w_1, w_2, w_3 \} \]

Now, \( w_3, w_4, w_5, \) and \( w_6 \) are all members of \( \mathbb{C} \). So, by definition, those four possible worlds are also members of \( (\mathbb{B} \cup \mathbb{C}) \). For, by definition, any member of \( \mathbb{C} \) also goes into the set \( (\mathbb{B} \cup \mathbb{C}) \). Thus, we know that \( (\mathbb{B} \cup \mathbb{C}) \) contains at least these six worlds:

\[ (\mathbb{B} \cup \mathbb{C}) = \{ w_1, w_2, w_3, w_4, w_5, w_6 \} \]

Note that we already knew that \( w_3 \) went into the set \( (\mathbb{B} \cup \mathbb{C}) \) because it was a member of \( \mathbb{B} \). We do not add it again once we realize that \( w_3 \) is also a member of \( \mathbb{C} \). That is, we do not do this:

\[ (\mathbb{B} \cup \mathbb{C}) = \{ w_1, w_2, w_3, w_3, w_4, w_5, w_6 \} \]

The reason is that, once something is in a set, it is in that set. Something cannot be in a set twice. Finally there is nothing else to add to the set \( (\mathbb{B} \cup \mathbb{C}) \). For we have already added all the members of \( \mathbb{B} \), and we have already added all the members of \( \mathbb{C} \), and, by definition, nothing else goes into the set \( (\mathbb{B} \cup \mathbb{C}) \). So we know that:

\[ (\mathbb{B} \cup \mathbb{C}) = \{ w_1, w_2, w_3, w_4, w_5, w_6 \} \]

Now, so far, we have only been talking about the union of one set \( S_1 \) with a single other set \( S_2 \). But we can also define the union of a whole bunch of sets \( S_1, \ldots, S_n \) with each other. The basic idea is exactly the same. The union of \( S_1, \ldots, S_n \) = the set that results when you combine, or unite, all of the members of \( S_1 \), with all of the members of \( S_2 \), with all of the members of \( S_3 \), and so on, until you finally add all the members of \( S_n \), and then stop.
So, for instance, consider these four sets:

\[ A = \{w_1, w_2, w_3\} \]
\[ B = \{w_3, w_4, w_5\} \]
\[ C = \emptyset \]
\[ D = \{w_5, w_6, w_7, w_8\} \]

When we add all of the members of \( A \), together with all of the members of \( B \), together with all of the members of \( C \), together with all of the members of \( D \), and then stop, we get this set: \( \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\} \). So that set is the union of \( A, B, C, \) and \( D \).

To refer to the union of a whole series of sets \( S_1, \ldots, S_n \) with each other we write: \( (S_1 \cup \cdots \cup S_n) \). So, for example, to refer to the union of \( A, B, C, \) and \( D \) we write: \( (A \cup B \cup C \cup D) \).

\[
(S_1 \cup \cdots \cup S_n) = \text{the union of } S_1, \ldots, S_n = \text{the set that results when you combine, or unite, all of the members of } S_1, \text{ with all of the members of } S_2, \text{ with all of the members of } S_3, \text{ and so on, until you finally add all the members of } S_n, \text{ and then stop. That is, by definition:}
\]

\[
\begin{align*}
&\text{• if } x \in S_1, \text{ then } x \in (S_1 \cup \cdots \cup S_n) \\
&\text{• if } x \in S_2, \text{ then } x \in (S_1 \cup \cdots \cup S_n) \\
&\text{• if } x \in S_3, \text{ then } x \in (S_1 \cup \cdots \cup S_n) \\
&\text{• \ldots} \\
&\text{• if } x \in S_n, \text{ then } x \in (S_1 \cup \cdots \cup S_n) \\
&\text{• Otherwise, } x \notin (S_1 \cup \cdots \cup S_n)
\end{align*}
\]

We can also use some visual representations to help us grasp what the union of some sets is. So, for example, suppose that we use the circles below to represent the sets \( S_1, S_2, \) and \( S_3 \):
The union of $S_1$ with $S_2$, that is $(S_1 \cup S_2)$, is the set that results when you combine all of the members of $S_1$, together with all of the members of $S_2$, and then stop. So the union of those two sets is represented by the shaded area in this diagram:

By contrast, the union of all three sets, that is $(S_1 \cup S_2 \cup S_3)$, is the set that results when you combine all of the members of $S_1$, together with all of the members of $S_2$, together with all of the members of $S_3$, and then stop. So the union of those three sets is represented by the shaded area in this diagram:
10.5 Intersection

The intersection of a set $S_1$ with a set $S_2$ is itself a further set. It is the set that results when you take all of the things that are members of both $S_1$ and $S_2$, and put them into a further set, and then stop. So, for instance, consider these two sets of possible worlds:

$$F = \{w_1, w_2\}$$
$$G = \{w_2, w_7\}$$

The intersection of $F$ with $G$ is the set that results when you take all of the things that are members of both $F$ and $G$, and put them into a further set, and then stop. For $w_2$ is the only thing that is a member of both $F$ and $G$.

To refer to the intersection of a set $S_1$ with a set $S_2$ we write: $(S_1 \cap S_2)$. We will sometimes leave off the parentheses for the sake of convenience. So, for instance, officially, to refer to the intersection of the set $F$ with the set $G$ we would write: $(F \cap G)$. But often, as long as there is no chance of ambiguity or confusion, we will simply write: $F \cap G$.

$$(S_1 \cap S_2) = \text{the intersection of } S_1 \text{ with } S_2 = \text{the set that results when you take all of the things that are members of both } S_1 \text{ and } S_2, \text{ put them into a further set, and then stop. That is, by definition:}$$

- if $x \in S_1$ and $x \in S_2$, then $x \in (S_1 \cap S_2)$
- Otherwise, $x \notin (S_1 \cap S_2)$

So consider again these two sets of possible worlds:

$$M = \{w_1, w_2, w_5\}$$
$$K = \{w_3, w_4\}$$

Note that, while there are some things that are members of $M$, and there are some things that are members of $K$, there is not anything which is a member of both $M$ and $K$. So there is not anything which is a member of $(M \cap K)$. The intersection of those two sets is empty. That is, $(M \cap K) = \emptyset$.

Apart from talking about the intersection of one set with another, we can also talk about the intersection of a whole series of sets $S_1, \ldots, S_n$ with each other. The basic idea is the same. The intersection of a whole bunch of sets $S_1, \ldots, S_n$ is the
set that results when you take all of the things that are members of every set in the series, put them into a new set, and then stop.

So, for instance, consider these four sets of possible worlds:

\[
\begin{align*}
A &= \{w_1, w_2, w_3\} \\
B &= \{w_3, w_4, w_5\} \\
C &= \{w_1, w_3, w_4\} \\
D &= \{w_3, w_6, w_7\}
\end{align*}
\]

Note that \(w_3\) is the only thing that is a member of all four sets. So the set that results when you take the things that are members of each and every set above, put them into a new set, and then stop, is this: \(\{w_3\}\). So that set is the intersection of the sets \(A, B, C,\) and \(D\).

To refer to the intersection of a whole series of sets \(S_1, \ldots, S_n\) with each other we write: \((S_1 \cap \cdots \cap S_n)\). So, for example, to refer to the intersection of \(A, B, C,\) and \(D\) we write: \((A \cap B \cap C \cap D)\).

\[
(S_1 \cap \cdots \cap S_n) = \text{the intersection of } S_1, \ldots, S_n = \text{the set that results when you take the things that are members of every set in the series } S_1, \ldots, S_n, \text{ put them into a new set, and then stop. That is, by definition:}
\]

- if \(x \in S_1\), and \(x \in S_2\), and \(x \in S_3\), \ldots, and \(x \in S_n\), then \(x \in (S_1 \cap \cdots \cap S_n)\)
- Otherwise, \(x \notin (S_1 \cap \cdots \cap S_n)\)

We can again use some visual representations to help us grasp what the intersection of some sets is. So, for example, suppose that we use the circles below to represent the sets \(S_1, S_2,\) and \(S_3\):
The intersection of $S_1$ and $S_2$, that is $(S_1 \cap S_2)$, is represented by the shaded area in this diagram:

\[ S_1 \cap S_2 \]

By contrast, the intersection of all three sets, that is $(S_1 \cap S_2 \cap S_3)$, is represented by the shaded area in this diagram:

\[ S_1 \cap S_2 \cap S_3 \]

Do Practice Questions E
A.1 Consider the sentence ‘Colin plays basketball’. Suppose that we replace the word ‘plays’ in that sentence with the word ‘coaches’. Does the new sentence communicate the same information as the original sentence? In other words, is the proposition expressed the same? Answer Yes or No.

A.2 Consider the sentence ‘Bradley has two sisters’. Suppose that we replace the word ‘sisters’ in that sentence with the phrase ‘female siblings’. Does the new sentence make the same claim about the world as the original sentence? In other words, is the proposition expressed the same? Answer Yes or No.

A.3 Consider the sentence ‘Both John and Amanda like horror movies’. Now construct a new sentence as follows: first, take the sentence ‘John likes horror movies’, and add the word ‘and’ to the very end of it; second, after the word ‘and’, tack on the sentence ‘Amanda likes horror movies too’. Does the new sentence express the same information as the original sentence? Answer Yes or No.

A.4 Consider these two sentences: first, ‘John is twelve inches from the edge of the cliff”; second, ‘John is one foot from the edge of the cliff”. Do these sentences make the very same claim about the world? Answer Yes or No.

A.5 Consider the sentence ‘Mars is more than one million miles from Earth’. Suppose that we replace the word ‘Mars’ with the word ‘Jupiter’. Does the new sentence communicate the same piece of information as the original?

A.6 Consider the sentence ‘Mars is more than one million miles from Earth’. Suppose that we replace the occurrence of ‘Mars’ at the beginning of the sentence with the word ‘Earth’, and then we replace the word ‘Earth’ at the end of the sentence with the word ‘Mars’. Does the new sentence express the same proposition as the original one? That is, do the two sentences communicate exactly the same piece of information? Answer Yes or No.

A.7 Consider the sentence ‘Hillary Clinton will win the 2016 Presidential Election’. Suppose that we stick the phrase ‘Everyone believes that’ in front of that sentence. Yes or No: Does the new sentence make the same claim about the world as the original?
A.8 Consider these three sentences: first, ‘Marge and Homer are both Red Sox fans’; second, ‘Marge is a Red Sox fan’; third, ‘Homer is a Red Sox fan’. Answer Yes or No to each of the following questions: Do the first and second sentences express the same piece of information? Do the first and third? If we create a fourth sentence by sticking the word ‘and’ between the second and the third sentences, would the first and fourth sentences express the same proposition?

A.9 Consider these three sentences: first, ‘At least one of Smith and Jones is a liar’; second, ‘Smith is a liar’; third, ‘Jones is a liar’. Answer Yes or No to each of the following questions: Do the first and second sentences make the same claim about the world? Do the first and third? Suppose we create a fourth sentence in the following way: we write the word ‘Either’; then we insert the second sentence; then we write the word ‘or’; then we insert the third sentence; and then, finally, we insert the phrase ‘or both are’. Do the first and fourth sentences express the same proposition?

A.10 Consider these three sentences: first, ‘John owns a car’; second, ‘John does not own a car’; third, ‘It is not the case that John does not own a car’. Answer Yes or No to both of the following questions: Do the first and second sentences express the same piece of information? Do the first and third?

A.11 Consider these two sentences: first, ‘Neither Frank nor Maria was killed by a zombie’; second, ‘Frank was not killed by a zombie and Maria was not killed by a zombie’. Answer Yes or No: Do these two sentences express the same proposition?

A.12 Consider these two sentences: first, ‘With regard to Frank and Maria, at least one of the two was killed by a zombie’; second, ‘It is not the case that neither Frank nor Maria was killed by a zombie’. Yes or No: Do these two sentences make exactly the same claim about the world? In other words, is the proposition expressed the same in both cases?

A.13 Consider these two sentences: first, ‘Not all of them were eaten by wolves’; second, ‘None of them were eaten by wolves’. Answer Yes or No: Do these two sentences express the same piece of information?

A.14 Consider these two sentences: first, ‘Not all of them were eaten by wolves’; second, ‘At least one of them was not eaten by wolves’. Answer Yes or No: Do these two sentences express the same proposition, or claim about the world?
A.15 Consider these two sentences: first, ‘At least one of Frank and Maria is infected’; second, ‘It is not the case that both Frank and Maria are not infected’. Yes or No: Do these two sentences express the same proposition?

B

B.1 Using the table below, indicate, for each proposition that is listed, whether it is true or false at the worlds in question:

<table>
<thead>
<tr>
<th></th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
<th>$p_5$</th>
<th>$p_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>@</td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$w_1$</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Propositions:
- $p_1 =$ that there is a 25,000 ft. tall mountain in Massachusetts
- $p_2 =$ that Tom Brady has won a Super Bowl
- $p_3 =$ that Barack Obama wrote a book titled *Dreams from My Father*
- $p_4 =$ that Barack Obama was born in Hawaii
- $p_5 =$ that there are 47 states in the United States of America
- $p_6 =$ that Al Gore won the 2000 presidential election

Worlds:
- @ = the actual world
- $w_1 =$ a possible world where the Supreme Court does not stop the recount that was under way in Florida during the 2000 election and Al Gore wins that election; during his presidency, he trades 3 states—Oklahoma, Oregon, and Ohio—to Nepal, leaving the US with 47 states overall. In exchange, Nepal agrees to make Mt. Everest (which is over 25,000 ft. tall) an official part of the state of Massachusetts. Barack Obama, after finishing his book *Dreams from My Father* in early 2004, while on vacation in his native Hawaii, decides to pursue mountain climbing rather than politics. Meanwhile, several members of the Jets, scheduled to play the Patriots early in the 2000 season,
die in a plane crash, and their game with the Patriots is canceled. Drew Bledsoe, not having to face the Jets defense, remains healthy for that season and several to come. Tom Brady remains on the bench, and never wins a Super Bowl.

B.2 Using the table below, indicate, for each proposition that is listed, whether it is true or false at the worlds in question:

<table>
<thead>
<tr>
<th></th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
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<th>$p_5$</th>
<th>$p_6$</th>
<th>$p_7$</th>
<th>$p_8$</th>
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<tbody>
<tr>
<td>$w_1$</td>
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<td>$w_3$</td>
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<td></td>
</tr>
<tr>
<td>$w_4$</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Propositions:

- $p_1$ = that John is hungry
- $p_2$ = that Mary is hungry
- $p_3$ = that John is not hungry
- $p_4$ = that both John and Mary are hungry
- $p_5$ = that neither John nor Mary is hungry
- $p_6$ = that either John, or Mary, or both are hungry
- $p_7$ = that it is not the case that both John and Mary are hungry
- $p_8$ = that either John is not hungry, or Mary is not hungry, or both are not hungry

Worlds:

- $w_1$ = a world where both John and Mary are quite hungry.
- $w_2$ = a world where John is very hungry, but Mary is not hungry at all.
- $w_3$ = a world where John is not hungry, but where Mary is hungry
- $w_4$ = a world where John is not hungry at all, and Mary is not hungry at all either
C

C.1 Suppose that Greg says, “I’m hungry,” and Isaac says, “I’m hungry too.” Both Greg and Isaac thereby make a claim about the world. That is, both express or put forward some proposition. Let’s let:

- \( p \) = the proposition that Greg expresses when he says, “I’m hungry”
- \( q \) = the proposition that Isaac expresses when he says, “I’m hungry too”
- \( w \) = a world where Greg has not had anything to eat since breakfast, and is starving as a result, but where Isaac just ate five large chocolate chip cookies, and so is quite full

Answer each of the following questions: (i) Do Greg and Isaac utter the exact same sentence? (ii) \( p \) is true / false at \( w \) (circle the right answer) (iii) \( q \) is true / false at \( w \) (circle the right answer) (iv) Are \( p \) and \( q \) identical to each other? That is, are they one and the same proposition? (v) When Greg says “I’m hungry,” and Isaac says, “I’m hungry too,” do they make the very same claim about the world?

C.2 Suppose that Amelia is in Dugan Hall, whereas Tyler is in Coburn Hall. Each utters the sentence, ‘There is gold buried underneath this building’. By uttering that sentence, each expresses a proposition. Let:

- \( p \) = the proposition that Amelia expresses when she says, standing in Dugan Hall, “There is gold buried underneath this building”
- \( q \) = the proposition that Tyler expresses when he says, standing in Coburn Hall, “There is gold buried underneath this building”
- \( w \) = a world where there are no precious metals at all buried underneath Dugin Hall, but there are substantial gold deposits present underneath Coburn.

Answer each of the following questions: (i) \( p \) is true / false at \( w \) (circle the right answer) (ii) \( q \) is true / false at \( w \) (circle the right answer) (iii) Does \( p = q \)? That is, are \( p \) and \( q \) one and the same proposition? (iv) Do Amelia and Tyler make the very same claim about the world? (v) Do Amelia and Tyler utter exactly the same sentence?

C.3 Suppose that Catherine, in the year 1990, says “The Red Sox won exactly one World Series within the last 10 years.” Then, in the year 2010, Monique says, “The
Red Sox won exactly one World Series within the last 10 years.” Each makes a certain claim about the world. That is, each expresses a certain proposition. Let:

- \( p = \) the proposition that Catherine expresses when she says, in the year 1990, “The Red Sox won exactly one World Series within the last 10 years”
- \( q = \) the proposition that Monique expresses when she says, in the year 2010, “The Red Sox won exactly one World Series within the last 10 years”
- \( w = \) a world where the Red Sox do not even make it to the playoffs throughout the entire 1980s and 1990s, but then finally breakthrough in 2005 and go on to win the World Series that year. Unfortunately, they do not win the World Series again until 2015.

Answer each of the following questions: (i) \( p \) is true / false at \( w \) (circle the right answer) (ii) \( q \) is true / false at \( w \) (circle the right answer) (iii) Does \( p = q \)? (iv) Do Catherine and Monique make the very same claim about the world? (v) Do Catherine and Monique utter exactly the same sentence?

C.4 Suppose that Mark says, “China has over 10 million people,” and Andy says, “India has over 10 million people”. Let:

- \( p = \) the proposition that Mark expresses when he says, “China has over 10 million people”
- \( q = \) the proposition that Andy expresses when he says, “India has over 10 million people”
- \( @ = \) the actual world

Answer each of the following questions: (i) \( p \) is true / false at \( @ \) (circle the right answer) (ii) \( q \) is true / false at \( @ \) (circle the right answer) (iii) Do \( p \) and \( q \) have the same truth-value at \( @ \)? (iv) Is there any possible world at which \( p \) and \( q \) differ in truth-value—that is, where one is true and the other is false? (v) Does \( p = q \)?

C.5 Suppose that George says, “Hillary Clinton is the President of the US,” and Kate says, “Joe Biden is the President of the US”. Let:

- \( p = \) the proposition that George expresses when he says, “Hillary Clinton is the President of the US”
• \( q \) = the proposition that Kate expresses when she says, “Joe Biden is the President of the US”

• \( @ \) = the actual world

Answer each of the following questions: (i) \( p \) is true / false at \( @ \) (circle the right answer) (ii) \( q \) is true / false at \( @ \) (circle the right answer) (iii) Do \( p \) and \( q \) have the same truth-value at \( @ \)? (iv) Is there any possible world at which \( p \) and \( q \) differ in truth-value? (v) Does \( p = q \)?

D

D.1 True or False: A restaurant is in Massachusetts if it is in Boston.

D.2 True or False: A restaurant is in Massachusetts only if it is in Boston.

D.3 True or False: A person counts as an employee of the federal government if he or she is a Senator.

D.4 True or False: A person counts as an employee of the federal government only if he or she is a Senator.

D.5 True or False: A student is likely to be admitted to Harvard if he or she graduated high school.

D.6 True or False: A student is likely to be admitted to Harvard only if he or she graduated high school.

D.7 Suppose we know that a person is eligible to practice law in Massachusetts if he or she passes the Massachusetts state bar exam. That is, suppose we know that, to be eligible to practice law in Massachusetts, it is sufficient to pass the state bar exam. Suppose we also know that John has passed the exam. Can we conclude that John is eligible to practice law in Massachusetts? Answer Yes or No. What if we find out that Clara has not passed the Massachusetts bar exam? Can we then conclude for certain that Clara is not eligible to practice law in Massachusetts? Answer Yes or No.

D.8 Let’s assume that a keyboard is useful only if its space bar works. Suppose that the space bar on Aaron’s keyboard does not work. Does it follow from our assumption that Aaron’s keyboard is not useful? Answer Yes or No. Let’s further
suppose that the space bar on Juliet’s keyboard does work. Does it then follow from our assumption that Juliet’s keyboard is useful? Answer Yes or No.

D.9 True or False: A proposition is a special kind of sentence.

D.10 True or False: Someone who looks out her window in San Francisco and says, “It’s raining,” expresses the very same proposition as someone who looks out her window in Boston and says, “It’s raining.”

D.11 True or False: There is a possible world at which Barack Obama has three arms.

D.12 True or False: There is a possible world at which Barack Obama was never given the name ‘Barack Obama’.

D.13 Here is a table showing the truth-values of propositions $p$ and $q$ at five different possible worlds:

<table>
<thead>
<tr>
<th></th>
<th>$p$</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$w_2$</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$w_3$</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>$w_4$</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>$w_5$</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Does $p = q$?

D.14 Here is a table showing the truth-values of propositions $p$ and $q$ at worlds $w_1$–$w_4$:

<table>
<thead>
<tr>
<th></th>
<th>$p$</th>
<th>$q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>$w_2$</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>$w_3$</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>$w_4$</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Let us pretend that worlds $w_1$–$w_4$ are the only possible worlds that there are. Does $p = q$?
Let us suppose that there are just four possible worlds: $w_1, w_2, w_3,$ and $w_4$. Let us further suppose that propositions $p$, $q$, and $r$ have the following truth-values at those four worlds:

\[
\begin{array}{ccc}
  & p & q & r \\
 w_1 & T & T & F \\
 w_2 & F & T & T \\
 w_3 & T & F & F \\
 w_4 & F & F & T \\
\end{array}
\]

E.1 Let $P_t$ = the set of possible worlds at which $p$ is true. Fill in the members of $P_t$.

$P_t = \{ \}$

E.2 Let $P_f$ = the set of possible worlds at which $p$ is false. Fill in the members of $P_f$.

$P_f = \{ \}$

E.3 $(P_t \cap P_f) =$ ?

a. $\{w_1\}$

b. $\{w_3, w_4\}$

c. $\emptyset$

E.4 Let $Q_t$ = the set of possible worlds at which $q$ is true, and let $Q_f$ = the set of possible worlds at which $q$ is false. $(Q_t \cap Q_f) =$ ?

a. $\{w_1, w_2\}$

b. $\emptyset$

c. $\{w_3, w_4\}$

E.5 Let $R_t$ = the set of possible worlds at which $r$ is true, and let $R_f$ = the set of possible worlds at which $r$ is false. $(R_t \cap R_f) =$ ?

a. $\emptyset$
b. \( \{w_2, w_3, w_4\} \)  
c. \( \{w_1, w_2, w_3, w_4\} \)

E.6 Fill in the members of \((P_t \cup Q_t)\):

\[(P_t \cup Q_t) = \{ \} \]

E.7 Fill in the members of \((P_t \cup Q_t \cup R_t)\):

\[(P_t \cup Q_t \cup R_t) = \{ \} \]

E.8 Fill in the members of \((P_f \cap Q_f)\):

\[(P_f \cap Q_f) = \{ \} \]

E.9 Fill in the members of \((P_f \cap Q_f \cap R_f)\):

\[(P_f \cap Q_f \cap R_f) = \{ \} \]

E.10 True or False: \( w_2 \in (P_t \cup Q_f) \)

E.11 Fill in the members of \(((P_t \cup Q_t) \cap R_f)\):

\[((P_t \cup Q_t) \cap R_f) = \{ \} \]

E.12 Fill in the members of \((Q_f \cap R_t \cap P_f)\):

\[(Q_f \cap R_t \cap P_f) = \{ \} \]

E.13 Fill in the members of \((P_t \cup Q_t) \cap (P_f \cup Q_f)\):

\[(P_t \cup Q_t) \cap (P_f \cup Q_f) = \{ \} \]

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E.14 Let $S$ be some set. $S \cup \emptyset = ?$
   a. $\emptyset$
   b. $S$

E.15 Let $S$ be some set. $S \cap \emptyset = ?$
   a. $\emptyset$
   b. $S$

E.16 $\mathbb{P}_t$ is a subset of __________:
   a. $\mathbb{Q}_t$
   b. $\mathbb{R}_f$
   c. $(\mathbb{R}_t \cup \mathbb{R}_f)$
   d. both b. and c.

E.17 Fill in the members of the sets $A$ and $B$ so that, $(A \cap B) = \emptyset$, yet $(A \cup B) = \{w_1, w_2, w_3, w_4\}$.
   $$A = \{\}$$
   $$B = \{\}$$

E.18 Consider the set of propositions $\{p, q, r\}$. __________ is a possible world at which all of the propositions in that set are true:
   a. $w_1$
   b. $w_2$
   c. $w_3$
   d. $w_4$
   e. none of the above
E.19 Consider the set of propositions \( \{p, q\} \). __________ is a possible world at which both of the propositions in that set are true and at which \( r \) is false:

a. \( w_2 \)

b. \( w_1 \)

c. \( w_3 \)

d. none of the above

E.20 True or False: \( P_t = R_t \).

E.21 Let \( K_t = \{w_1, w_2\} \). In the table below, fill in the column for \( k \). Note that every proposition is either true or false at any given world. So if a proposition is not true at a certain world, then it is false there.

<table>
<thead>
<tr>
<th></th>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>( w_2 )</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>( w_3 )</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>( w_4 )</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
</tr>
</tbody>
</table>

E.22 \( k = ? \)

a. \( p \)

b. \( q \)

c. \( r \)

d. none of the above

E.23 Let \( M_t = \{w_1, w_3\} \). \( m = ? \)

a. \( r \)

b. \( p \)

c. \( q \)

d. none of the above
E.24 Let $J_f = \text{the set of possible worlds at which } j \text{ is false} = \mathbb{R}_f$. Add a column for $j$ to the above table, and fill it in.

E.25 Let $J_t = \text{the set of possible worlds at which } j \text{ is true}$. Fill in the members of this set:

$$J_t = \{ \}$$

E.26 True or False: $j = p$.

E.27 True or False: $j = r$. 

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